Lecture 3. Solvability of linear systems

Prop Consider a linear system represented by a matrix A.

(1) If the last column of RREF(A) contains a leading 1, the system has no solutions.

$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad 0 = 1 \Longrightarrow \text{no solutions}$$

(2) If all columns except the last column of RREF(A) contain a leading 1, the system has a unique solution.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

The solution is given by the last column

(3) In all other cases, the system has infinitely many solutions.

$$\left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Columns without a leading 1 yield free variables

Ex For each linear system, determine the number of its solutions.

(1) $\begin{cases} 2x_1 \qquad\qquad -4x_4 = 2 \\ \qquad\qquad x_3 - x_4 = 3 \\ 3x_1 + 2x_2 \qquad\qquad = 5 \end{cases}$

Sol We simplify the matrix of the system using row operations.

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & -4 & 2 \\ 0 & 0 & 1 & -1 & 3 \\ 3 & 2 & 0 & 0 & 5 \end{array}\right] \xrightarrow[\frac{1}{2}R_1]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 & 3 \\ 3 & 2 & 0 & 0 & 5 \end{array}\right] \xrightarrow[R_3 - 3R_1]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 2 & 0 & 6 & 2 \end{array}\right]$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 1 \\ 0 & 2 & 0 & 6 & 2 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right] \xrightarrow[\frac{1}{2}R_2]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right] \quad \text{RREF!}$$

The last column does not contain a leading 1.

Column 4 does not contain a leading 1.

$\Rightarrow$ The system has $\boxed{\text{infinitely many solutions}}$

Note We have to check the last column before checking the other columns for free variables.

e.g. A linear system represented by the matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

has no solutions.

(2) $\begin{cases} x_1 - 2x_2 \qquad\quad + 2x_4 = 1 \\ -2x_1 + 4x_2 + 3x_3 - 7x_4 = 4 \\ 3x_1 - 6x_2 - 2x_3 + 8x_4 = 6 \end{cases}$

Sol We simplify the matrix of the system using row operations.

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 1 \\ -2 & 4 & 3 & -7 & 4 \\ 3 & -6 & -2 & 8 & 6 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 + 2R_1} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & 3 & -3 & 6 \\ 0 & 0 & -2 & 2 & 3 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & -2 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{R_3 + 2R_2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 7 \end{array}\right]$$

The last row yields the equation $0 = 7$ which never holds.

$\Rightarrow$ The system has $\boxed{\text{no solutions}}$

Note (1) As in this example, we can often determine whether a linear system is solvable before computing the RREF of its matrix.

(2) The final matrix

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 7 \end{array}\right]$$

is an example of an echelon form where zeros form a staircase pattern in the bottom left corner.

Ex Find the unknown constant a such that the linear system

$$\begin{cases} x_1 + 2x_2 + x_3 = a \\ x_2 + 3x_3 = 2 \\ x_1 \quad - 5x_3 = 3 \end{cases}$$

has a solution.

Sol We simplify the matrix of the system using row operations.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 3 & 2 \\ 1 & 0 & -5 & 3 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 3 & 2 \\ 0 & -2 & -6 & 3-a \end{array}\right] \xrightarrow{R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 7-a \end{array}\right]$$

The last row yields the equation $0 = 7 - a \Longrightarrow \boxed{a = 7}$

Note In fact, for $a = 7$ we get the RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and thus get infinitely many solutions of the system.

(with column 3 yielding a free variable)